DYNAMICS OF UNIDIRECTIONAL GLASS-FIBER-REINFORCED PLASTIC

A. M. Mikhailov

UDC 539.31

The motion problem of unidirectional glass-fiber-reinforced plastic is formulated under the assumption that the fibers are under stress – strain only, while the binder is under shear stress only. The binder and fiber inertia is calculated along a direction parallel to the fibers. The system of equations in partial derivatives obtained is reduced by Laplace transformation with respect to time to a system of ordinary differential equations in which only the fiber displacements occur. As illustration, the effect of a normal stress wave on a half space is solved. The solution is obtained in the form of an infinite series provided with an explicit law by which the terms are obtained. Curves are presented for the distribution of the normal and shearing stresses at different moments of time. The binder inertia reduces to the appearance of tangential stresses at the fiber – binder boundary, which can explain the tendency towards stratification in constructions made of glass fiber-reinforced plastic.

1. The assumption [1] that normal stresses exist only in reinforced fibers and that tangential stresses exist only in the binder in areas parallel to the fibers is often used in studying equilibrium of plates made of unidirectional glass-fiber-reinforced plastic.

Such a theoretical treatment of the nature of the performance of the components is justified by the fact that Young's moduli differ in them by 1-2 times, while stretches are roughly the same due to the cohesion of the fiber and binder. Although the stress state of the components of glass-fiber-reinforced plastic is in fact more complex, such an approach correctly expresses the concept of the efficient performance of reinforced material: high strength fibers are oriented along the tensile stress lines and the binder facilitates a more uniform distribution of these loads between the fibers.

The mathematical formulation of static problems leads from this standpoint to a system of ordinary differential equations for the displacements of reinforced fibers, while elasticity theory would lead to a more complex problem formulated in terms of equations in partial derivatives with coupling conditions on each fiber – binder surface.

Such an approach is generalized below to the case of dynamics. Here, we will preserve the assumptions formulated regarding the nature of the performance of the components and take into account inertial forces in both the binder and in the fibers. This leads to a system of partial differential equations for the displacement. A system of differential equations is obtained following Laplace transformation with respect to time and the elimination of the binder displacements for the transformed fiber displacements, almost identical to the static case.

The chief qualitative effect clarified by our formulation of the problem is that the binder inertia induces an increase in the shearing stresses at the component interfaces.

Suppose a given plate consists of M fibers with Young's modulus E and density ρ_1, the fibers numbered with integers j from R + 1 to R + M. The fibers alternate with binder layers (shear modulus G, density ρ_2) of width H. We denote by h the width of the fibers in the place of the plate. Thus we have for the reinforcement coefficient $\psi = h\,(h + H)$, the velocity of the shear waves in the fibers $c_1 = (E/\rho_1)^{1/2}$, the velocity of the shear waves in the binder $C_2 = (G/\rho_2)^{1/2}$. The y axis is parallel, and the x axis perpendicular, to the fibers.

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 139-145, July-August, 1974. Original article submitted March 26, 1974.

Binder displacement along the y axis will be noted by $v_j(\xi, y, t)$. This notation indicates a given binder point lies between the j-th and (j + 1)-th fibers at a distance ξ from the j-th fiber ($0 \le \xi \le H$); y is the coordinate along the fiber, and t is time. Fiber displacement will be denoted by $u_j(y, t)$. We assume that shearing stresses in the binder $\tau_j(\xi, y, t)$ and normal stresses in the fiber $\sigma_j(y, t)$ are proportional to the corresponding deformations,

$$\tau_j(\xi, y, t) = G\frac{\partial v_j(\xi, y, t)}{\partial \xi}, \quad \sigma_j(y, t) = E\frac{\partial u_j(y, t)}{\partial y} \tag{1.1}$$

Then the motion equation of the binder changes into the wave equation

$$c_2{}^2 \partial^2 v_i / \partial \xi^2 - \partial^2 v_i / \partial t^2 = 0 \tag{1.2}$$

The binder–fiber coupling conditions yield the boundary conditions for Eq. (1.2):

$$v_j(0, y, t) = u_j(y, t), \quad v_j(H, y, t) = u_{j+1}(y, t) \tag{1.3}$$

Let us write the fiber motion equations, assuming that the outermost fibers (j + R + 1, j = R + M) are free of external loads, so that

$$h\partial\sigma_j/\partial y + [(1 - \delta_{j,R+M})\,\tau_j(0, y, t) - (1 - \delta_{j,R+1})\,\tau_{j-1}(H, y, t)] = \rho_1 h \partial^2 u_j / \partial t^2$$

where δ_{jk} is the Kronecker symbol. After substituting σ_j, τ_j, and τ_{i-1} from Eq. (1.1) we obtain

$$\frac{\partial^2 u_j}{\partial y^2} + \frac{G}{Eh}\frac{\partial}{\partial \xi}[(1 - \delta_{j,R+M})\, v_j|_{\xi=0} - (1 - \delta_{j,R+1})\, v_{j-1}|_{\xi=H}] = \frac{1}{c_1{}^2}\frac{\partial^2 u_j}{\partial t^2}, \quad c_1{}^2 = \frac{E}{\rho_1} \tag{1.4}$$

The system of equations (1.2)-(1.4) determines, in conjunction with the corresponding initial and boundary conditions, the motion of glass-fiber-reinforced plastic.

We now Laplace transform Eqs. (1.2)-(1.4) with respect to time, assuming for the sake of simplicity that the initial conditions are zero ([2] Chap. VI). The transformed variables will be marked with the superscript L and the transformation parameter will be denoted by p. Following this tranformation Eq. (1.2) changes into an ordinary differential equation and, after solving it under the boundary conditions (1.3) (Laplace transformed), we obtain

$$\begin{gathered} v_j{}^L(\xi, y, p) = \mathrm{sh}^{-1}\lambda\,[u_j{}^L \mathrm{sh}(\lambda - \lambda\xi / H) + u_{j+1}{}^L \mathrm{sh}(\lambda\xi / H)] \\ \lambda = pH / c_2 \end{gathered} \tag{1.5}$$

Using Eqs. (1.4) and (1.5) we arrive at the system of ordinary differential equations

$$\begin{gathered} \omega^2 d^2 u_{R+1}{}^L / dy^2 + \beta^2[-(\alpha - \mathrm{ch}\,\lambda)\,u_{R+1}{}^L + u_{R+2}{}^L] = 0 \\ \omega^2 d^2 u_j{}^L / dy^2 + \beta^2(u_{j-1}{}^L - \alpha u_j{}^L + u_{j+1}{}^L) = 0, \; j \neq R+1, R+M \\ \omega^2 d^2 u_{R+M}^L / dy^2 + \beta^2[u_{R+M-1}^L - (\alpha - \mathrm{ch}\lambda)\,u_{R+M}^L] = 0 \\ \beta = (G/E)^{1/2}, \quad \omega = (Hh\,\mathrm{sh}\,\lambda/\lambda)^{1/2}, \quad \alpha = p^2\omega^2 / \beta^2 c_1 + 2\mathrm{ch}\,\lambda \end{gathered} \tag{1.6}$$

Fiber displacements but not binder displacements occur in Eq. (1.6). Under nonzero initial conditions a known right side would appear in Eq. (1.6). The shearing stresses in the binder are determined by means of Eqs. (1.5) and (1.1) once u_j has been found.

Let us consider the limiting cases. Suppose the displacements approach a given limit as t tends to infinity. Then, using a well-known theorem of the operational calculus ([2], section 83), we have

$$\lim_{t\to\infty} u_j = \lim_{p\to 0} p u_j{}^L$$

Multiplying Eqs. (1.5) and (1.6) by t and passing to the limit as p approaches zero, we obtain the system of equations that describes the equilibrium state. It will differ from Eqs. (1.6) by the absence of L superscripts for the displacements u_j and by the fact that $\lambda = 0$, $\omega = (Hh)^{1/2}$, and that $\alpha = 2$. We obtain from Eq. (1.5) following this passage to a limit that the displacement equilibrium of the binder are linear functions of ξ.

We may arrive at the same results by letting the transmission rates c_1 and c_2 of the interaction tend toward infinity. By passing to the limit, the coefficients of the system (1.6) tend to the same values as when p approaches zero. Consequently, the fundamental system of solutions (1.6) will be the same as in statics. Thus, the dependence of u_j on p results by definition from the boundary conditions on the constants occurring in the general solution of the system. That is, glass-fiber-reinforced plastic instantaneously reacts to a variation in the boundary conditions, acquiring an equilibrium configuration corresponding to the boundary conditions at the given moment of time. If, in particular, the boundary displacements or deformations are constant when $t > 0$, their Laplace transforms (transformation parameter p^{-1}) following the transformation of the preimage u_j will turn out to be independent of t ([2], sec. 79, Eq. (8) when $\tau = 0$).

Let us consider a second passage to a limit. We let h and H approach zero and the number M of fibers tend to infinity, so that the reinforcement coefficient $\psi = h/(h + H)$ and the $x = j(h + H)$ remain constant. Then α approaches 2 and the expression $u_{j-1}^L - \alpha u_j^L + u_{j+1}$ in the second and third equations of Eq. (1.6) becomes a proportional difference analogue of the second derivative with respect to x and u^L.

Thus we pass to a homogeneous continuous medium, the equation for which in Laplace transforms is obtained from Eq. (1.6):

$$\frac{\partial^2 u^L}{\partial y^2} + \frac{\beta^2}{\psi(1-\psi)} \frac{\partial^2 u^L}{\partial x^2} = \frac{p^2}{c_1^2} u^L$$

An equation in preimages corresponds to this equation, that is

$$\frac{\partial^2 u}{\partial y^2} + \frac{\beta^2}{\psi(1-\psi)} \frac{\partial^2 u}{\partial x^2} = \frac{1}{c_1^2} \frac{\partial^2 u}{\partial t^2} \tag{1.7}$$

which little differs from a wave equation. The binder inertia following this passage to a limit does not play any role. Consequently, a description of glass-fiber-reinforced plastic by means of Eq. (1.7) is unacceptable whenever the dynamic interaction between the fiber and binder is of interest.

2. Let us consider the half-space $y \geq 0$, $-\infty < j < \infty$. In this case, only the second and third equations remain in Eqs. (1.6): multiplying the j-th equation by $\exp(isj)$, where i is imaginary unity and s is a real number and summing over j, we arrive at the ordinary equation

$$\omega^2 d^2 u^{LF} / dy^2 + \beta^2 (-\alpha + 2\cos s) u^{LF} = 0$$

for the Fourier series

$$u^{LF}(y, p, s) = \sum_{j=-\infty}^{\infty} u_j^L(y, p) \exp(isj)$$

Since $u^{LF}(\infty, p, s) = 0$, we find

$$u^{LF}(y, p, s) = c(p, s) \exp\left(-\frac{\beta y}{\omega} \sqrt{\alpha - 2\cos s}\right)$$

The unknown function $c(p, s)$ is determined from the boundary conditions when $y = 0$. We find by expressing the coefficients of the Fourier series in terms of its sum, the Laplace transform of the displacements

$$u_j^L(y, p) = (2\pi)^{-1} \int_{-\pi}^{\pi} u^{LF}(y, p, s) \exp(-isj)\, ds$$

Suppose suddenly applied constant normal stress $Q\psi$ (ψ is the reinforcement coefficient) acts on the half space.

Under the model we have accepted (the binder does not absorb normal leads) the boundary conditions are given by

$$\frac{\partial u_j}{\partial y}(0, t) = \frac{Q}{E} \delta_0(t), \qquad \delta_0(t) = \begin{cases} 0, & t \leqslant 0 \\ 1, & t > 0 \end{cases}$$

Following Laplace transformation we have

$$\frac{\partial u_j^L}{\partial y}(0, p) = \frac{Q}{pE}, \quad \frac{\partial u^{LF}}{\partial y}(0, p, s) = \frac{2\pi Q}{pE} \sum_{k=-\infty}^{\infty} \delta_1(s - 2\pi k)$$

where δ_1 is the Dirac delta function ([3], p. 47, Eq. (2)),

$$c(p, s) = -\frac{2\pi Q \omega}{\beta \sqrt{\alpha - 2\cos s}} \sum_{k=-\infty}^{\infty} \delta_1(s - 2\pi k)$$

$$u_j^L(y, p) = -\frac{\omega Q}{p\beta E} \frac{\exp(-\beta y \sqrt{\alpha - 2}/\omega)}{\sqrt{\alpha - 2}}$$

$$\sigma_j^L(y, p) = E \frac{du_j^L}{dy} = -\frac{Q}{p} \exp\left(-\frac{yr}{c_1}\right)$$

$$r = \sqrt{\varepsilon p \operatorname{th}\frac{\lambda}{2} + p^2}, \qquad \varepsilon = \frac{2Gc_1^2}{Ehc_2}$$

We find the shearing stresses at the fiber–binder boundary ($\xi = 0$) using Eqs. (1.1) and (1.5):

$$\tau_j^L(0, y, p) = -\frac{Qc_1}{\beta^2 c_2 r} \operatorname{th}\frac{\lambda}{2} \exp\left(-\frac{yr}{c_1}\right)$$

Since there is no dependence in this problem on j, we omit this subscript in subsequent calculations.

To obtain the preimages, we decompose τ^L and σ^L in Taylor series in powers $\mu = \exp(-\lambda)$ in the neighborhood of the point $\mu = 0$. The terms of these series correspond to the shear waves due to the most distant fibers, based on the lag theorem of the operational calculus ([4], p. 69). In particular, the free term ($\lambda = \infty$) indicates the solution of the problem for an isolated fiber embedded in a half-space made of the binder or the solution of our problem for those moments of time when the shear wave can reach the neighboring fibers ($0 \leq t < H$).

In this case, differentiation with respect to μ or a calculation of the coefficients of the Taylor series can be replaced by differentiation with respect to the parameter ε, in which p does not occur.

In fact, introducing a new variable $\lambda = \varepsilon \tanh(\lambda/2)$ we have

$$\frac{\partial^n}{\partial \varepsilon^n} = \left(\operatorname{th}\frac{\lambda}{2}\right)^n \frac{\partial^n}{\partial \gamma^n}$$
$$\frac{\partial^n}{\partial \mu^n} = \sum_{k=1}^{n} \sum_{l=0}^{k-1} \frac{(-1)^l}{k!} C_k{}^l \gamma^l \frac{\partial^n \gamma^{k-l}}{\partial \mu^n} \frac{\partial^k}{\partial \gamma^k} \tag{2.1}$$

The second equality expresses the rule for finding the n-th derivatives of a composite function ([5], Eq. (0.430)]. Using the first equality and recalling that the derivatives must be found at the point $\mu = 0$, and also that

$$\gamma = \varepsilon(1-\mu)/(1+\mu)$$
$$\sum_{l=0}^{k-1} (-1)^l C_k{}^l \gamma^l \frac{\partial^n}{\partial \mu^n} \gamma^{k-l}\Big|_{\mu=0} = \frac{\partial^n}{\partial \mu^n} \sum_{l=0}^{k-1} (-1)^l C_k{}^l \varepsilon^l \gamma^{k-l}\Big|_{\mu=0} =$$
$$= \varepsilon^k \frac{\partial^n}{\partial \mu^n}\left(\frac{1-\mu}{1+\mu} - 1\right)^k\Big|_{\mu=0} = (-2\varepsilon)^k (-1)^{n-k} k(n-1)!\, C_n{}^k$$

we find the n-th term of the desired Taylor series for the normal stress

$$\frac{(-1)^n \mu^n}{n} \sum_{k=1}^{n} \frac{(2\varepsilon)^k}{(k-1)!} C_n{}^k \frac{\partial^k \sigma^L}{\partial \varepsilon^k} (\mu = 0) \tag{2.2}$$

Since the parameter ε is independent of p, it will be sufficient to find the preimage for the free term, while all the remaining preimages corresponding to $n \geq 1$ will be obtained from it by differentiation with respect to ε, a shift with respect to time by nH/c_2 in the expression obtained and substitution in Eq. (2.2). We then obtain the normal stress using the transformation equation [6] as a function of time and coordinate;

$$\frac{\sigma(y,t)}{Q} = \delta_0\left(t - \frac{y}{C_1}\right)\left[\exp\left(-\frac{\varepsilon y}{2c_1}\right) + \frac{\varepsilon y}{2c_1} \int_{y/c_1}^{t} \frac{\exp(-\varepsilon\alpha/2)\, I_1\left(\sqrt{\alpha^2 - y^2/c_1^2}\right) d\alpha}{\sqrt{\alpha^2 - y^2/c_1^2}}\right] -$$
$$- \frac{\varepsilon y}{2c_1} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \delta_0\left(t - \frac{nH}{c_2} - \frac{y}{c_1}\right) \sum_{k=1}^{n} C_n{}^k \frac{(2\varepsilon)^{k-1}}{(k-1)!} \frac{\partial^{k-1}}{\partial \varepsilon^{k-1}}\left[\exp\left(-\frac{\varepsilon}{2}\left(t - \frac{nH}{c_2}\right)\right) I_0\left(\frac{\varepsilon}{2}\sqrt{t_n^2 - \frac{y^2}{c_1^2}}\right)\right],$$

where I_0 and I_1 are Bessel functions of an imaginary argument.

Similar calculations yield the shearing stress at the fiber bonudary

$$-\frac{\tau(y,t)\, c_2}{Q\beta^2 c_1} = \exp\left(-\frac{\varepsilon t}{2}\right) I_0\left(\frac{\varepsilon}{2}\sqrt{t^2 - \frac{y^2}{c_1^2}}\right) \delta_0\left(t - \frac{y}{c_1}\right) +$$
$$+ \sum_{n=1}^{\infty} \frac{(-1)^n}{n\varepsilon} \delta_0\left(t_n - \frac{y}{c_1}\right) \sum_{k=1}^{n} \frac{(2\varepsilon)^k}{(k-1)!} C_n{}^k \frac{\partial^k}{\partial \varepsilon^k} \left[\varepsilon \exp\left(-\frac{\varepsilon t_n}{2}\right) I_0\left(\frac{}{2}\sqrt{t_n^2 - \frac{y^2}{c_1^2}}\right)\right] \tag{2.4}$$

At any time t a finite number of terms will occur in Eqs. (2.3) and (2.4) so that the Heaviside function vanishes for sufficiently large n. The physical meaning here is that a fiber interacts always with a finite number of other fibers, from which the shear wave succeeds in arriving at a given moment in time.

Equations (2.3) and (2.4) are not suitable for calculations on a computer as they involve the differentiation operation. We can eliminate differentiation by using recursion formulas for the Bessel functions. The result of such a transformation is as follows:

$$\frac{\sigma(\eta, T)}{Q} = \delta_0(T - \eta)\left[\exp(-\eta) + \eta \int_{\eta}^{T} \frac{\exp(-\alpha)\, I_1\left(\sqrt{\alpha^2 - \eta^2}\right) d\alpha}{\sqrt{\alpha^2 - \eta^2}}\right] - \eta \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \delta_0(T - n\varkappa - \eta) \sum_{k=1}^{n} q_{k-1}(T - n\varkappa, \eta)$$

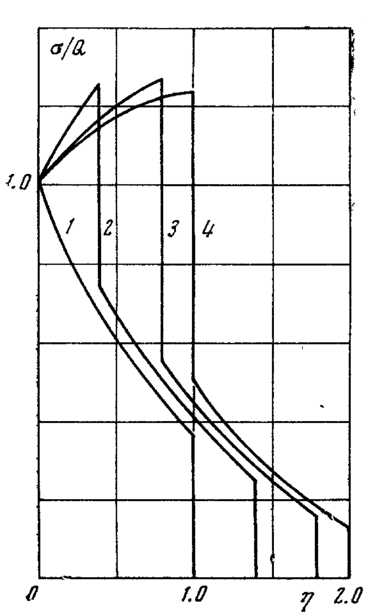

Fig. 1

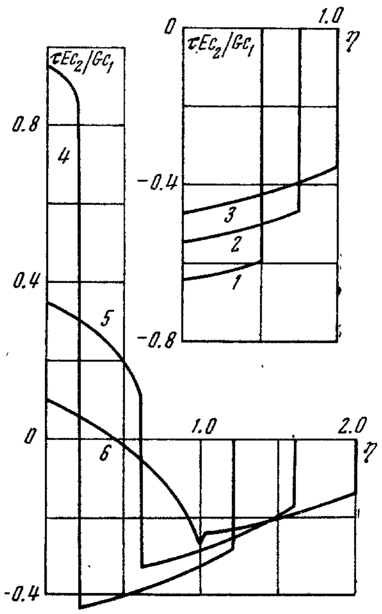

Fig. 2

$$-\frac{\tau(\eta, T)\,c_2}{Q\beta^2 c_1} = q_0(\eta, T)\,\delta_0(T-\eta) + \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\times\delta_0(T-n\varkappa-\eta)\times$$

$$\times\sum_{k=1}^{n} k\left[q_k(T-n\varkappa, \eta)+\frac{2}{k}(n-k+1)\,q_{k-1}(T-n\varkappa, \eta)\right]$$

where $\eta = \varepsilon y/2c_1 = \beta^2 c_1 y/c_2 h$ is a dimensionless coordinate, $T = \varepsilon t/2 = t\beta^2 c_1^2/c_2 h$ is dimensionless time, $\varkappa = \varepsilon H/2c_2 = \beta^2 c_1^2 H/c_2^2 h$, is dimensionless time for the shear wave path between the fibers, and

$$q_0(T, \eta) = \exp(-T)\,I_0(\sqrt{T^2-\eta^2})$$
$$m_0(T, \eta) = \exp(-T)\,I_1(\sqrt{T^2-\eta^2})$$
$$q_k(T, \eta) = \frac{2(n-k+1)}{k^2}\left(-Tq_{k-1}+\sqrt{T^2-\eta^2}\,m_{k-1}\right)$$
$$m_k(T, \eta) = \frac{n-k+1}{k^2}\left(\sqrt{T^2-\eta^2}\,q_{k-1}-Tm_{k-1}\right)+\frac{(-2)^k}{k}C_n^{\,k}\sum_{l=0}^{k-1}\frac{m_l}{(-2)^l C_n^{\,l}}$$

Figures 1 and 2 depict the stress distribution with respect to coordinate at different moments of time (the calculation was conducted with $\varkappa = 1$). If binder inertia is not taken into account, it turns out that $\tau = 0$ and $\sigma = Q\delta_0(c_1 t - y)$. Curves 1-4 correspond to T = 1.0, 1.4, 1.8, and 2.0 in Fig. 1 and curves 1-6 in Fig. 2 correspond to T = 0.5, 0.75, 1.0, 1.2, 1.6, and 2.0, respectively.

The qualitative effect demonstrated here, namely the appearance of shearing stresses in areas perpendicular to the front of a plane wave, is related to the binder inertia and not to the fact that the load at y = 0 is distributed nonuniformly (concentrated only on the fibers). If we take into account the normal rigidity of the binder and load the half-plane boundary uniformly, we will again arrive at the same problem with the only difference that the boundary conditions will have to hold on a line shifted inside the half-space by a value on the order of $c_1 h/c_2$, due to the lesser velocity of longitudinal waves in the binder than in the fibers over a period of time roughly equal to the shear wave path time between the fibers.

LITERATURE CITED

1. J. O. Outwater, Jr., "The mechanics of plastics reinforcement in tension," Mod. Plastics, 33, No. 7, 156 (1956).
2. M. A. Lavrent'ev and B. V. Shabam, Methods of the Theory of Functions of a Complex Variable [in Russian], Nauka, Moscow (1973).
3. I. M. Gel'fand and G. E. Shilov, Generalized Functions and Functional Operations [in Russian], Fizmatgiz, Moscow (1958).

4. L. I. Slepyan, Nonstationary Elastic Waves [in Russian], Sudostroenie, Leningrad (1972).
5. I. S. Gradshtein and I. M. Ryzhik, Tables of Integrals, Sums, Series, and Products [in Russian], Fizmatgiz, Moscow (1963).
6. V. A. Ditkin and A. P. Prudnikov, Handbook of Operational Calculus [in Russian], Vysshaya Shkola, Moscow (1965).